Sylver Coinage
An Algebraist's Investigation

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Overview

Sylver Coinage is a two-player game in which the players alternate choosing natural numbers.

In any turn the player cannot name any number that can be represented as a linear combination of the previously named numbers using non-negative coefficients. The player that must choose the number 1 loses.

In this talk, we investigate what is known about Sylver Coinage as well as how the study of numerical semigroups can inform our strategy.

Overview

Sylver Coinage Game

Sylver Coinage is a two-player game in which the players alternate choosing natural numbers

In any turn, the player cannot name any number that can be represented as a linear combination of the previously named numbers using non-negative coefficients

The player that must choose the number 1 loses

Example

Player 1: 10

$$M = \{10\}$$

$\mathbf{0}, 1, 2, 3, 4, 5, 6, 7, 8, 9, \mathbf{10}, 11, 12, 13, 14, 15, 16, 17, 18, 19, \mathbf{20}, 21, 22, 23, 24, 25, \cdots$

Example

Player 2: 5

$$M = \{10, 5\}$$

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, 7, 8, 9, \mathbf{10}, 11, 12, 13, 14, \mathbf{15}, 16, 17, 18, 19, \mathbf{20}, 21, 22, 23, 24, \mathbf{25}, \cdots$

Example

Player 1: 7

$$M = \{10, 5, 7\}$$

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

Example

Player 2: 23

$$M = \{10, 5, 7, 23\}$$

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Example

Player 1: 6

$$M = \{10, 5, 7, 23, 6\}$$

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, \mathbf{6}, \mathbf{7}, 8, 9, \mathbf{10}, \mathbf{11}, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Example

Player 2: 4

$$M = \{10, 5, 7, 23, 6, 4\}$$

$\mathbf{0}, 1, 2, 3, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Example

Player 1: 3

$$M = \{10, 5, 7, 23, 6, 4, 3\}$$

$\mathbf{0}, 1, 2, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Example

Player 2: 2

$$M = \{10, 5, 7, 23, 6, 4, 3, 2\}$$

$$\mathbf{0}, 1, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \mathbf{10}, \mathbf{11}, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$$

Player 2 Wins!

Numerical Semigroups

A subset S of $\mathbb{N}$ closed under addition, containing zero, and having a largest integer not in S

OR

The set S of all non-negative coefficient linear combinations of a set of generators whose greatest common divisor is 1

Example

Player 1: 7

$$M = \{10, 5, 7\}$$

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

Example

$$S = \langle 5, 7 \rangle$$

$$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$$

Definitions

Numerical Semigroups and Sylver Coinage have very different terminology for the same properties

Moves/Generators

$$M = \{10, 5, 7\} \text{ Moves}$$
$$S = \langle 5, 7 \rangle \text{ Generators}$$

M - Position, a set of moves
S - Numerical Semigroup, a set of linear combinations

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

$\mathcal{N}$ and $\mathcal{P}$

$\mathcal{N}$ - a position M where the next player can win
$\mathcal{P}$ - all other positions, previous to $\mathcal{N}$

g.c.d.

$$M = \{10, 5, 7\}$$
$$S = \langle 5, 7 \rangle$$

g - $g.c.d.(M)$
Note: $g = 1$ for Numerical Semigroup

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

Frobenius Number

$$M = \{10, 5, 7\}$$
$$S = \langle 5, 7 \rangle$$

$t\,(M)$ - If $g.c.d\,(M) = 1$, $t\,(M)$ is the greatest legal move
$F\,(S)$ - greatest number not in S

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

End

$$M = \{10, 5, 7, 23\}$$
$$S = \langle 5, 7, 23 \rangle$$

End - all moves x that do not eliminate any other move

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Pesudo-Frobenius Number

$$M = \{10, 5, 7, 23\}$$
$$S = \langle 5, 7, 23 \rangle$$

$PF(S)$ - set of all integers x such that $x \notin S$ and $x + s \in S$

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Ender

$$M = \{10, 5, 7, 23, 16\}$$
$$S = \langle 5, 7, 16 \rangle$$

Ender - A position M where $t(M)$ is the only end
S is irreducible, i.e. pesudo-symmetric or symmetric
$PF(S) = \{F(S)/2, F(S)\}$ or $\{F(S)\}$
Enders are $\mathcal{N}$, allow for strategy stealing

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

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Quiet Ender

$$M = \{10, 5, 7\}$$
$$S = \langle 5, 7 \rangle$$

Quiet Ender - Ender and every move eliminates $t(M)$ without multiples $PF(S) = \{F(S)\}$, S is symmetric

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

Quiet Ender

$$\text{M} = \{10, 5, 7\}$$
$$\text{S} = \langle 5, 7 \rangle$$

Quiet Ender - Ender and every move eliminates $t\left(M\right)$ without multiples
$PF\left(S\right) = \{F\left(S\right)\}$, S is **symmetric**
M displays Strong **Antisymmetery** Principle

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

Quiet Ender

$$\mathrm{M} = \{10, 5, 7\}$$
$$\mathrm{S} = \langle 5, 7 \rangle$$

Quiet Ender - Ender and every move can eliminate $t(M)$ without multiples
$PF(S) = \{F(S)\}$, S is symmetric
M displays Strong Antisymmetery Principle

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, 9, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, 16, \mathbf{17}, 18, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, 23, \mathbf{24}, \mathbf{25}, \cdots$

Strategies

Some strategies are known

Research in Numerical Semigroups provides other strategies for winning

Known Strategies

1. If $p \geq 5$, then $\{p\}$ is $\mathcal{P}$
2. If $p \geq 5$, then $\{ap\}$ loses to p
3. If M is nontrivial and $g \geq 5$ is prime, then g wins
4. If $a, b \in \mathbb{Z}^+$, then $\{2^a 3^b\}$ is generally unknown

Enclosure

$$\mathrm{M} = \{10, 5, 7, 23, 16, 9\}$$
$$\mathrm{S} = \langle 5, 7, 9, 16 \rangle$$

$E\,(M)$ - Position obtained by imposing antisymmetry on M
$E\,(M) = \{10, 5, 7, 23, 16, 9, 11\}$

$\mathbf{0}, 1, 2, 3, 4, \mathbf{5}, 6, \mathbf{7}, 8, \mathbf{9}, \mathbf{10}, 11, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Strategies

1. Completing $E(M)$ results in $\mathcal{N}$
2. $F(S)$ is not always the winning move with an ender

Playing $F(S)$

$$\mathrm{M} = \{8, 7, 4\}$$
$$\mathrm{S} = \langle 4, 7 \rangle$$

Picking $F(S) = 17$ will result in loss

$\mathbf{0}, 1, 2, 3, \mathbf{4}, 5, 6, \mathbf{7}, \mathbf{8}, 9, 10, \mathbf{11}, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, \mathbf{16}, 17, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Playing $F(S)$

$$\text{M} = \{8, 7, 4, 17\}$$
$$\text{S} = \langle 4, 7, 17 \rangle$$

Picking $F(S) = 17$ will result in loss

$\mathbf{0}, 1, 2, 3, \mathbf{4}, 5, 6, \mathbf{7}, \mathbf{8}, 9, 10, \mathbf{11}, \mathbf{12}, 13, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Playing $F(S)$

$$M = \{8, 7, 4, 17, 13\}$$
$$S = \langle 4, 7, 13, 17 \rangle$$

Picking $F(S) = 17$ will result in loss
Player 2 picks 13
Player 1 cannot make a winning move

$\mathbf{0}, 1, 2, 3, \mathbf{4}, 5, 6, \mathbf{7}, \mathbf{8}, 9, 10, \mathbf{11}, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \mathbf{24}, \mathbf{25}, \cdots$

Strategies

1. Completing $E(M)$ results in $\mathcal{N}$
2. $F(S)$ is not always the winning move with an ender
3. Let m is the smallest move and e is the number of moves
4. If $S = \langle p, q \rangle$, then $S = \langle p, q, nq - p \rangle$ will be symmetric if $n|p$

Strategies

1. Completing $E(M)$ results in $\mathcal{N}$
2. $F(S)$ is not always the winning move with an ender
3. If $S = \langle p, q \rangle$, then $S = \langle p, q, nq - p \rangle$ will be symmetric if $n|p$
4. Let m is the smallest move and e is the number of independent moves
    1. If $2 \leq e \leq m - 1$, then there exists a irreducible numerical semigroup with these values of m and e
    2. Always need to be aware of how close semigroup is to irreducible

Future Questions

Some of Richard Guy's questions:

1. Is there an effective technique for computing the status of any M?
2. Is there an effective technique for producing good replies?
3. What is the status of position $\{n\}$ for n of the form 2^a3^b?
4. What is the status of $\{16\}$?
5. What is the status of $\{18\}$?
6. Is M a $\mathcal{P}$-position whenever $2M$ is a $\mathcal{P}$-position?
7. If the game is played "between intelligent players" is it always the case that the first person to make the game bounded is the loser?

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