The Frobenius Number of Balanced Numerical Semigroups

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Overview

We will discuss the algebraic structure known as a numerical semigroup and definitions related to them.

We will investigate the Frobenius number of balanced numerical semigroups.

Overview

Coin Exchange Problem

Given 5 and 7 cent coins, what is the largest amount of change that cannot be created?

$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \cdots$

Coin Exchange Problem

Given 5 and 7 cent coins, what is the largest amount of change that cannot be created?

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$$23 = 7 \cdot 5 - 7 - 5$$

Numerical Semigroups

A subset S of $\mathbb{N}$ closed under addition, containing zero, and having a largest integer not in S

$\mathbf{0}, 1, 2, 3, 4, 5, 6, 7, \mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, 15, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \cdots$

Numerical Semigroups

Let $a_1, a_2, ..., a_n \in \mathbb{N}$ such that $gcd\,(a_1, a_2, ..., a_n) = 1$.

$$S = \langle a_1, a_2, ..., a_n \rangle = \{a_1 t_1 + ... + a_n t_n | t_i \in \mathbb{N}\}$$
$$x \in S = (t_1, t_2, \cdots, t_n)$$

$\mathbf{0}, 1, 2, 3, 4, 5, 6, 7, \mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, 15, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \cdots$

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Let $a_1, a_2, ..., a_n \in \mathbb{N}$ such that $gcd\,(a_1, a_2, ..., a_n) = 1$.

$$S = \langle a_1, a_2, ..., a_n \rangle = \{a_1 t_1 + ... + a_n t_n | t_i \in \mathbb{N}\}$$
$$x \in S = (t_1, t_2, \cdots, t_n)$$

$a_1, a_2, ..., a_n$ are called generators

$\mathbf{0}, 1, 2, 3, 4, 5, 6, 7, \mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, 15, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \cdots$

Frobenius Number

Frobenius Number: largest integer not in S

$$S = \langle 8, 9, 11, 12 \rangle$$

$$F(S) = 15$$

$\mathbf{0}, 1, 2, 3, 4, 5, 6, 7, \mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \cdots$

Known Results

"It would be nice to have a finite set of formulas which covers all possible cases for computing $F(S)$. Unfortunately, no such collection of formulas has ben found and probably does not exist at all."

- 'The exact solutions to the Frobenius problem with three variables' by Mihaly Hujter and Bela Vizvri; J. Ramanujan Math. Soc. 2 (1987)

Known Results

- 2 Generators - $S = \langle a, b \rangle$ (Sylvester 1884)
  $F(S) = a \cdot b - a - b$
- 3+ Generators - $S = \langle a_1, a_2, \cdots, a_n \rangle$
  $F(S) =$???

Known Results

- 2 Generators - $S = \langle a, b \rangle$ (Sylvester 1884)
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- 3+ Generators - $S = \langle a_1, a_2, \cdots, a_n \rangle$
  $F(S) = ???$
- Arithmetic sequence $S = \langle a, ma + d, ma + 2d, \cdots, ma + kd \rangle$ (Lewin 1975)
  $F(S) = m \cdot a \cdot \left(1 + \lfloor \frac{a-2}{k} \rfloor\right) + (d-1)(a-1) - 1$
- $S =< a_1, \ldots, a_n >$ (not necessarily in ascending order) and let $d = gcd\{a_1, \ldots, a_{n-1}\}$ (Brauer and Schockley 1962)
  $F(S) = d \cdot F(\langle \frac{a_1}{d}, \ldots, \frac{a_{n-1}}{d}, a_n \rangle) + (d-1)\, a_n$
- $S =< a_1, a_2, a_3 >$ and $a_i | (a_1 + a_2 + a_3)$ for some $i = 2, 3$ (Brauer and Schockley 1962)
  $F(S) = -a_1 + max_{i=2,3}\{a_i \cdot \lfloor \frac{a_1 \cdot a_{5-i}}{a_2 + a_3} \rfloor\}$

Apery Set Relative to a_1

Apery Set relative to a_1:

All elements in S such that
$x \in S$ and $x - a_1 \notin S$

Apery Set Relative to a_1

Apery Set relative to a_1:

All elements in S such that
$x \in S$ and $x - a_1 \notin S$

or: all elements in S that can only be made without a_1

Note: $Max(Ap(S)) - a_1 = F(S)$

Apery Set Relative to a_1

Apery Set Relative to a_1 (8):
$x \in S$ and $x - 8 \notin S$

$$S = \langle 8, 9, 11, 12 \rangle$$

$\mathbf{0}, 1, 2, 3, 4, 5, 6, 7, \mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, 15, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \cdots$

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$$S = \langle 8, 9, 11, 12 \rangle$$

$$Ap(S) = \{0, 9, 11, 12, 18, 21, 22, 23\}$$

$$|\mathbf{0}, 1, 2, 3, 4, 5, 6, 7|\mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, \mathbf{15}|\mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}| \cdots$$

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$$S = \langle 8, 9, 11, 12 \rangle$$

$$Ap(S) = \{0, 9, 11, 12, 18, 21, 22, 23\}$$

Note: $Max(Ap(S)) - 8 = 23 - 8 = 15 = F(S)$

$$|\mathbf{0}, 1, 2, 3, 4, 5, 6, 7|\mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, \mathbf{15}|\mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}| \cdots$$

Example

$$S = \langle 6, 8, 13, 15 \rangle$$

$$0, 1, 2, 3, 4, 5, \mathbf{6}, 7, \mathbf{8}, 9, 10, 11, 12, \mathbf{13}, 14, \mathbf{15}, 16, 17, 18, 19, 20, 21, 22, 23, \cdots$$

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Example

$$S = \langle 6, 8, 13, 15 \rangle$$

$$F(S) = 17$$

$$\mathbf{0}, 1, 2, 3, 4, 5, \mathbf{6}, 7, \mathbf{8}, 9, 10, 11, \mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{15}, \mathbf{16}, \color{red}{\mathbf{17}}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \cdots$$

Example

$$S = \langle 6, 8, 13, 15 \rangle$$

$$F(S) = 17$$

$$Ap(S) = \{0, 8, 13, 15, 16, 23\}$$

$$|\mathbf{0}, 1, 2, 3, 4, 5|\mathbf{6}, 7, \mathbf{8}, 9, 10, 11|\mathbf{12}, \mathbf{13}, \mathbf{14}, \mathbf{15}, \mathbf{16}, \mathbf{17}|\mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}| \cdots$$

Balanced Numerical Semigroups

$$S = \langle pD, pD + n, qD - n, qD \rangle$$

Balanced Numerical Semigroups

$$S = \langle pD, pD + n, qD - n, qD \rangle$$

$$p, q, D, n \in \mathbb{N}$$

$$p < q \text{ and } gcd\,(p, q) = 1$$
$$gcd\,(D, n) = 1$$

$$D \geq 1 \text{ and } 1 \leq n < \frac{p+q}{2} D$$

Example

$$S = \langle pD, pD + n, qD - n, qD \rangle$$

$p = 2, q = 3, D = 4$, and $n = 1$

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Example

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$$\mathbf{0}, 1, 2, 3, 4, 5, 6, 7, \mathbf{8}, \mathbf{9}, 10, \mathbf{11}, \mathbf{12}, 13, 14, \mathbf{15}, \mathbf{16}, \mathbf{17}, \mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22}, \mathbf{23}, \cdots$$

$$F(S) = 15$$

Frobenius Number of Balanced Numerical Semigroups

We use relationships among the generators to find elements of $Ap(S)$

If $x \in S$ and $x = t_1 a_1 + t_2 a_2 + t_3 a_3 + t_4 a_4$

We say $x = (t_1, t_2, t_3, t_4)$

$t_1, t_2, t_3, t_4 \geq 0$

Relations Among the Generators

$$S = \langle 8, 9, 11, 12 \rangle$$

$$\begin{aligned}
&(A) \qquad (3,0,0,0) = (0,0,0,2) \\
&(B) \qquad (1,0,0,1) = (0,1,1,0) \\
&(C_0) \qquad (2,0,1,0) = (0,3,0,0) \\
&(D_0) \qquad (3,1,0,0) = (0,0,3,0) \\
&(C_1) \qquad (1,0,2,0) = (0,2,0,1) \\
&(D_1) \qquad (2,2,0,0) = (0,0,2,1)
\end{aligned}$$

Relations Among the Generators

Note that

$$(A) \quad (3,0,0,0) = (0,0,0,2)$$

implies

if $x = (0,0,0,a) \in Ap(S)$ then $a < 2$

Members of $Ap(S)$

$$S = \langle 8, 9, 11, 12 \rangle$$

$$\begin{array}{ll}
(A) & (3,0,0,0) = (0,0,0,2) \\
 & \text{if } x = (0,0,0,a) \in Ap(S) \text{ then } a < 2 \\
 & \\
(B) & (1,0,0,1) = (0,1,1,0) \\
 & \\
(C_0) & (2,0,1,0) = (0,3,0,0) \\
 & \text{if } x = (0,c_0,0,0) \in Ap(S) \text{ then } c_0 < 3 \\
 & \\
(D_0) & (3,1,0,0) = (0,0,3,0) \\
 & \text{if } x = (0,0,d_0,0) \in Ap(S) \text{ then } d_0 < 3 \\
 & \\
(C_1) & (1,0,2,0) = (0,2,0,1) \\
 & \text{if } x = (0,c_1,0,1) \in Ap(S) \text{ then } c_1 < 2 \\
 & \\
(D_1) & (2,2,0,0) = (0,0,2,1) \\
 & \text{if } x = (0,0,d_1,1) \in Ap(S) \text{ then } d_1 < 2
\end{array}$$

Members of $Ap(S)$

$$S = \langle 8, 9, 11, 12 \rangle,\ F(S) = 15$$

(C_1) $(1, 0, 2, 0) = (0, 2, 0, 1)$
if $x = (0, c_1, 0, 1) \in Ap(S)$ then $c_1 < 2$

(D_1) $(2, 2, 0, 0) = (0, 0, 2, 1)$
if $x = (0, 0, d_1, 1) \in Ap(S)$ then $d_1 < 2$

Members of $Ap(S)$

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if $x = (0, c_1, 0, 1) \in Ap(S)$ then $c_1 < 2$

(D_1) $\quad (2, 2, 0, 0) = (0, 0, 2, 1)$
if $x = (0, 0, d_1, 1) \in Ap(S)$ then $d_1 < 2$

$$(0, 1, 0, 1) < (0, 0, 1, 1), \text{ so}$$

$$F(S) = (0, 0, 1, 1) - 8 = 11 + 12 - 8 = 15$$

Relations Among the Generators

$$(A) \qquad (q,0,0,0) = (0,0,0,p)$$

$$(B) \qquad (1,0,0,1) = (0,1,1,0)$$

$$(C_0) \qquad \left(\tfrac{ap+aq+n}{p} + k, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a, 0\right) = \left(0, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a, 0, 0\right)$$

$$(D_0) \qquad \left(q - \tfrac{ap+aq+n}{p} - k + p, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a - p, 0, 0\right) = \left(0, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a + p, 0\right)$$

$\ldots$

$$(C_{p-1}) \qquad \left(\tfrac{ap+aq+n}{p} + k - p + 1, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a + p - 1, 0\right) = \left(0, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a - p + 1, 0, p-1\right)$$

$$(D_{p-1}) \qquad \left(q - \tfrac{ap+aq+n}{p} - k + 1, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a - 1, 0, 0\right) = \left(0, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a + 1, p-1\right)$$

Frobenius Number

Smaller k Values:
$$k < p - \frac{ap+aq+n}{p} + \frac{n}{d}$$

$$Max(Ap(S)) = (0, 0, \delta_{p-1}, p-1)$$

$$F(S) = \delta_{p-1} a_3 + (p-1)\, a_4 - a_1$$
$$= (pq - p - q)\, D + (qD - n)\left(\frac{pD - pk}{p+q} - a\right)$$

Larger k Values:
$$k > p - \frac{ap+aq+n}{p} + \frac{n}{d}$$

$$Max(Ap(S)) = (0, \kappa_{p-1}, 0, p-1)$$

$$F(S) = \kappa_{p-1} a_2 + (p-1)\, a_4 - a_1$$
$$= (pq - p - q)\, D + (pD + n)\left(\frac{qD + pk}{p+q} + a - p\right)$$

Future Research

Future research:

Expand proof into smaller values of D

Develop new formula for very small values of D

Balanced numerical semigroups with $6, 8, 10$ or more generators

Questions

Questions?

Thanks

Research Partners:

Dr Kurt Herzinger, USAFA

Dr Trae Holcomb, Houston Baptist University

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k - Congruence Class of D Modulo $p+q$

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$$k \in \{1 - \tfrac{ap+aq+n}{p}, 2 - \tfrac{ap+aq+n}{p}, \ldots, p+q - \tfrac{ap+aq+n}{p}\}$$

where $a = -nq^{-1} \bmod p$

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$$\text{where } a = -nq^{-1} \bmod p$$

Example: $p = 2, q = 9, D = 8$, and $n = 1$

$$k \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$
$$k = -3$$

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We say $x = (t_1, t_2, t_3, t_4)$

$t_1, t_2, t_3, t_4 \geq 0$

Relations Among the Generators

Note that

$$(A) \quad (q, 0, 0, 0) = (0, 0, 0, p)$$

implies

if $x = (0, 0, 0, a) \in Ap(S)$ then $a < p$

Relations Among the Generators

$$
\begin{array}{ll}
(A) & (q,0,0,0) = (0,0,0,p) \\[1em]
(B) & (1,0,0,1) = (0,1,1,0) \\[1em]
(C_0) & \left(\frac{ap+aq+n}{p} + k, 0, \frac{pD}{p+q} - \frac{pk}{p+q} - a, 0\right) \\
& = \left(0, \frac{qD}{p+q} + \frac{pk}{p+q} + a, 0, 0\right) \\[1em]
(D_0) & \left(q - \frac{ap+aq+n}{p} - k + p, \frac{qD}{p+q} + \frac{pk}{p+q} + a - p, 0, 0\right) \\
& = \left(0, 0, \frac{pD}{p+q} - \frac{pk}{p+q} - a + p, 0\right) \\[1em]
\ldots & \\[1em]
(C_{p-1}) & \left(\frac{ap+aq+n}{p} + k - p + 1, 0, \frac{pD}{p+q} - \frac{pk}{p+q} - a + p - 1, 0\right) \\
& = \left(0, \frac{qD}{p+q} + \frac{pk}{p+q} + a - p + 1, 0, p - 1\right) \\[1em]
(D_{p-1}) & \left(q - \frac{ap+aq+n}{p} - k + 1, \frac{qD}{p+q} + \frac{pk}{p+q} + a - 1, 0, 0\right) \\
& = \left(0, 0, \frac{pD}{p+q} - \frac{pk}{p+q} - a + 1, p - 1\right)
\end{array}
$$

Relations Among the Generators

Note: These values only valid when

$$k \in \{p - \frac{ap+aq+n}{p}, \ldots, q - \frac{ap+aq+n}{p}\}$$

$$D \geq k + \frac{ap+aq}{p}, -\frac{kp}{q} - \frac{ap+aq}{q} + \frac{p}{q}(p+q)$$

Relations Among the Generators

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$$k \in \{p - \tfrac{ap+aq+n}{p}, \ldots, q - \tfrac{ap+aq+n}{p}\}$$

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Similar arguments needed for

$$k \in \{1 - \tfrac{ap+aq+n}{p}, \ldots, p - 1 - \tfrac{ap+aq+n}{p}\}$$

and $k \in \{q + 1 - \tfrac{ap+aq+n}{p}, \ldots, p + q - \tfrac{ap+aq+n}{p}\}$

Relations Among the Generators

Note: These values only valid when
$k \in \{p - \frac{ap+aq+n}{p}, \ldots, q - \frac{ap+aq+n}{p}\}$
$D \geq k + \frac{ap+aq}{p}, -\frac{kp}{q} - \frac{ap+aq}{q} + \frac{p}{q}(p+q)$

Similar arguments needed for
$k \in \{1 - \frac{ap+aq+n}{p}, \ldots, p - 1 - \frac{ap+aq+n}{p}\}$
and $k \in \{q + 1 - \frac{ap+aq+n}{p}, \ldots, p + q - \frac{ap+aq+n}{p}\}$

But ... results are the same for all values of k

Apery Set Relative to pD

Relations give us elements of $Ap(S)$

$$(A) \qquad (0,0,0,a),\ a \in \{0,\cdots,p-1\}$$

$$(C_0) \qquad (0,c_0,0,0),\ c_0 \in \{1,\cdots,\tfrac{qD}{p+q}+\tfrac{pk}{p+q}+a-1\}$$

$$(D_0) \qquad (0,0,d_0,0),\ d_0 \in \{1,\cdots,\tfrac{pD}{p+q}-\tfrac{pk}{p+q}-a+p-1\}$$

$$\cdots$$

$$(C_{p-1}) \qquad (0,c_{p-1},0,p-1),\ c_{p-1} \in \{1,\cdots,\tfrac{qD}{p+q}+\tfrac{pk}{p+q}+a-p\}$$

$$(D_{p-1}) \qquad (0,0,d_{p-1},p-1),\ d_{p-1} \in \{1,\cdots,\tfrac{pD}{p+q}-\tfrac{pk}{p+q}-a\}$$

$Max(Ap(S))$

$Max(Ap(S))$ is one of

$$
\begin{array}{ll}
(A) & (0,0,0,\alpha),\ \alpha = p-1 \\
(C_0) & (0,\kappa_0,0,0),\ \kappa_0 = \frac{qD}{p+q} + \frac{pk}{p+q} + a - 1 \\
(D_0) & (0,0,\delta_0,0),\ \delta_0 = \frac{pD}{p+q} - \frac{pk}{p+q} - a + p - 1 \\
\ldots & \\
(C_{p-1}) & (0,\kappa_{p-1},0,p-1),\ \kappa_{p-1} = \frac{qD}{p+q} + \frac{pk}{p+q} + a - p \\
(D_{p-1}) & (0,0,\delta_{p-1},p-1),\ \delta_{p-1} = \frac{pD}{p+q} - \frac{pk}{p+q} - a
\end{array}
$$

Frobenius Number

Smaller k Values:
$k < p - \frac{ap+aq+n}{p} + \frac{n}{d}$

$$Max(Ap(S)) = (0, 0, \delta_{p-1}, p-1)$$

$$F(S) = \delta_{p-1} a_3 + (p-1)\, a_4 - a_1$$
$$= (pq - p - q)\, D + (qD - n) \left(\frac{pD-pk}{p+q} - a \right)$$

Larger k Values:
$k > p - \frac{ap+aq+n}{p} + \frac{n}{d}$

$$Max(Ap(S)) = (0, \kappa_{p-1}, 0, p-1)$$

$$F(S) = \kappa_{p-1} a_2 + (p-1)\, a_4 - a_1$$
$$= (pq - p - q)\, D + (pD + n) \left(\frac{qD+pk}{p+q} + a - p \right)$$

D Ranges

Middle k Values:

$$k \in \{p - \tfrac{ap+aq+n}{p}, \ldots, q - \tfrac{ap+aq+n}{p}\}$$
$$D \geq k + \tfrac{ap+aq}{p}, -\tfrac{kp}{q} - \tfrac{ap+aq}{q} + \tfrac{p}{q}(p+q)$$

Smaller k Values:

$$k \in \{1 - \tfrac{ap+aq+n}{p}, \ldots, p - 1 - \tfrac{ap+aq+n}{p}\}$$
$$D \geq k + \tfrac{ap+aq}{p}, -\tfrac{kp}{q} - \tfrac{ap+aq}{q} + \tfrac{i}{q}(p+q)$$
$$i = k + \tfrac{ap+aq+n}{p}$$

Larger k Values:

$$k \in \{q + 1 - \tfrac{ap+aq+n}{p}, \ldots, p + q - \tfrac{ap+aq+n}{p}\}$$
$$D \geq k + \tfrac{ap+aq}{p} - \tfrac{p-i}{q}(p+q), -\tfrac{kp}{q} - \tfrac{ap+aq}{q} + \tfrac{p}{q}(p+q)$$
$$i = q + p - k - \tfrac{ap+aq+n}{p}$$

Problem - Smaller k

The relations $C_i, \cdots, C_{p-1}$ have invalid a_1 values

$$t_{C_i} = 0 \text{ for } i = \frac{ap+aq+n}{p} + k$$

Can repair with relation (B)

$$(1, 0, -1, 1) = (0, 1, 0, 0)$$

Relations Among the Generators - Smaller k

$$
\begin{aligned}
&(A) && (q,0,0,0)=(0,0,0,p)\\
&(C_0) && \left(\tfrac{ap+aq+n}{p}+k,0,\tfrac{pD}{p+q}-\tfrac{pk}{p+q}-a,0\right)\\
& && =\left(0,\tfrac{qD}{p+q}+\tfrac{pk}{p+q}+a,0,0\right)\\
&(D_0) && \left(q-\tfrac{ap+aq+n}{p}-k+p,\tfrac{qD}{p+q}+\tfrac{pk}{p+q}+a-p,0,0\right)\\
& && =\left(0,0,\tfrac{pD}{p+q}-\tfrac{pk}{p+q}-a+p,0\right)\\
&\ldots && \\
&(C_i^*) && \left(\tfrac{ap+aq+n}{p}+k-i+1,0,\tfrac{pD}{p+q}-\tfrac{pk}{p+q}-a+i-1,1\right)\\
& && =\left(0,\tfrac{qD}{p+q}+\tfrac{pk}{p+q}+a-i+1,0,i\right)\\
&\ldots && \\
&(C_{p-1}^*) && \left(\tfrac{ap+aq+n}{p}+k-i+1,0,\tfrac{pD}{p+q}-\tfrac{pk}{p+q}-a+i-1,p-i\right)\\
& && =\left(0,\tfrac{qD}{p+q}+\tfrac{pk}{p+q}+a-i+1,0,p-1\right)\\
&(D_{p-1}) && \left(q-\tfrac{ap+aq+n}{p}-k+1,\tfrac{qD}{p+q}+\tfrac{pk}{p+q}+a-1,0,0\right)\\
& && =\left(0,0,\tfrac{pD}{p+q}-\tfrac{pk}{p+q}-a+1,p-1\right)
\end{aligned}
$$

Apery Set - Smaller k

The Apery Set can be represented as before

But we have duplicates and invalid elements as below:

$$(C_i) \quad \left(0, 0, \tfrac{pD}{p+q} - \tfrac{k}{p+q} - a + i, j\right) = \left(0, \tfrac{qD}{p+q} + \tfrac{k}{p+q} + a - i, 0, i + j\right), j \in \{0, \cdots, p - i - 1\}$$

$$(C_{i+1}) \quad \left(0, 0, \tfrac{pD}{p+q} - \tfrac{k}{p+q} - a + i + 1, j\right) = \left(1, \tfrac{qD}{p+q} + \tfrac{k}{p+q} + a - i + 1, 0, i + 1 + j\right), j \in \{0, \cdots, p - i - 2\}$$

$\cdots$

$$(C_{p-1}) \quad \left(0, 0, \tfrac{pD}{p+q} - \tfrac{k}{p+q} - a + p - 1, j\right) = \left(p - i, \tfrac{qD}{p+q} + \tfrac{k}{p+q} + a - p + 1, 0, p - 1 + j\right), j \in \{0\}$$

Problem - Larger k

The relations $D_i, \cdots, D_{p-1}$ have invalid a_1 values

$$t_{D_i} = 0 \text{ for } i = q - \frac{ap+aq+n}{p} - k + p$$

Can repair with relation (B)

$$(1, -1, 0, 1) = (0, 0, 1, 0)$$

Relations Among the Generators - Larger k

$$(A) \qquad (q,0,0,0) = (0,0,0,p)$$

$$(C_0) \qquad \left(\tfrac{ap+aq+n}{p} + k, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a, 0\right) = \left(0, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a, 0, 0\right)$$

$$(D_0) \qquad \left(q - \tfrac{ap+aq+n}{p} - k + p, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a - p, 0, 0\right) = \left(0, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a + p, 0\right)$$

$\cdots$

$$(D_i^*) \qquad \left(q - \tfrac{ap+aq+n}{p} - k + p - i + 1, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a - p + i - 1, 0, 1\right) = \left(0, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a + p - i + 1, i\right)$$

$\cdots$

$$(C_{p-1}) \qquad \left(\tfrac{ap+aq+n}{p} + k - p + 1, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a + p - 1, 0\right) = \left(0, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a - p + 1, 0, p - 1\right)$$

$$(D_{p-1}^*) \qquad \left(q - \tfrac{ap+aq+n}{p} - k + p - i + 1, \tfrac{qD}{p+q} + \tfrac{pk}{p+q} + a - p + i - 1, 0, p - i\right) = \left(0, 0, \tfrac{pD}{p+q} - \tfrac{pk}{p+q} - a + p - i + 1, p - 1\right)$$

Apery Set - Larger k

The Apery Set can be represented as before

But we have duplicates and invalid elements as below:

$$(D_i) \quad \left(0, \tfrac{qD}{p+q} + \tfrac{k}{p+q} + a - p + i, 0, j\right) = \left(0, 0, \tfrac{pD}{p+q} - \tfrac{k}{p+q} - a + p - i, i + j\right), j \in \{0, \cdots, p - i - 1\}$$

$$(D_{i+1}) \quad \left(0, \tfrac{qD}{p+q} + \tfrac{k}{p+q} + a - p + i + 1, 0, j\right) = \left(1, 0, \tfrac{pD}{p+q} - \tfrac{k}{p+q} - a + p - i - 1, i + 1 + j\right), j \in \{0, \cdots, p - i - 2\}$$

$\cdots$

$$(D_{p-1}) \quad \left(0, \tfrac{qD}{p+q} + \tfrac{k}{p+q} + a - 1, 0, j\right) = \left(p - i, 0, \tfrac{pD}{p+q} - \tfrac{k}{p+q} - a + 1, p - 1 + j\right), j \in \{0\}$$

Frobenius Number

Smaller k Values:

$$k < p - \frac{ap+aq+n}{p} + \frac{n}{d}$$

$$Max(Ap(S)) = (0, 0, \delta_{p-1}, p-1)$$

$$F(S) = \delta_{p-1} a_3 + (p-1)\, a_4 - a_1$$
$$= (pq - p - q)\, D + (qD - n)\left(\frac{pD - pk}{p+q} - a\right)$$

Larger k Values:

$$k > p - \frac{ap+aq+n}{p} + \frac{n}{d}$$

$$Max(Ap(S)) = (0, \kappa_{p-1}, 0, p-1)$$

$$F(S) = \kappa_{p-1} a_2 + (p-1)\, a_4 - a_1$$
$$= (pq - p - q)\, D + (pD + n)\left(\frac{qD + pk}{p+q} + a - p\right)$$

The Frobenius Number of Balanced Numerical Semigroups

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