United States Air Force Academy

Numerical Semigroups and Wilf's Conjecture

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Summary

- We will discuss the algebraic structure known as a numerical semigroup and basic definitions related to them.
- We will examine Wilf's conjecture and an approach to a possible solution using intersections of symmetric semigroups.

Overview

- Definitions
- Example
- Wilf’s Conjecture
- Example
- My Investigation
- Open Questions

Definitions

Numerical Semigroup –

a subset S of $\mathbb{N}$ (the non-negative integers) closed under addition, containing zero, and having a largest integer not in S

S = the set of:

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 →…

Definitions

Embedding Dimension –

the minimal number of generators for the numerical semigroup, denoted by μ(S)

S = < *6*, *8*, *13* >

μ(S) = 3

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

Definitions

Frobenius Number –

the largest integer not contained in *S*, denoted by g(*S*)

$S = < 6, 8, 13 >$

$\mu(S) = 3$

$g(S) = 23$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

Definitions

Number of 'Small' Elements –

the number of elements in the numerical semigroup less than the Frobenius number, denoted by n(*S*)

$S = < 6, 8, 13 >$

$\mu(S) = 3$

$g(S) = 23$

$n(S) = 12$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

Definitions

Symmetric –

when the interval [0, g(*S*)] contains equally as many integers in *S* as outside of *S*

$S = \langle 6, 8, 13 \rangle$

$\mu(S) = 3$

$g(S) = 23$

$n(S) = 12$

S is symmetric

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

12 elements outside of *S* in [0, g(*S*)]

Example

$S = < 5, 7, 16 >$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

Example

$S = < 5, 7, 16 >$

$\mu(S) = 3$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

Example

$S = < 5, 7, 16 >$

$\mu(S) = 3$

$g(S) = 18$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

Example

$S = < 5, 7, 16 >$

$\mu(S) = 3$

$g(S) = 18$

$n(S) = 9$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

Example

$S = < 5, 7, 16 >$

$\mu(S) = 3$

$g(S) = 18$

$n(S) = 9$

S is pseudosymmetric

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

9 elements outside of S in $[\, 0, g(S) - 1 \,]$

9 elements inside of S in $[\, 0, g(S) - 1 \,]$

Example

Psudosymmetric –

when the interval [0, g(S) - 1] contains equally as many integers in S as outside of S

Equivalently, a numerical semigroup S with Frobenius number g(S) is symmetric/pseudosymmetric when S contains the maximum possible number of ‘small’ elements.

Example

$S = \langle 5, 7, 16 \rangle$

$\mu(S) = 3$

$g(S) = 18$

$n(S) = 9$

S is pseudosymmetric

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

9 elements outside of S in $[\ 0, g(S) - 1\ ]$

9 elements inside of S in $[\ 0, g(S) - 1\ ]$

SO… lets get on to the cool stuff already!!!

Wilf's Conjecture

Background –

In his paper *A Circle-Of-Lights Algorithm For The “Money-Changing Problem”* Dr. Herbert S. Wilf presented the following open question:

Is it always true that for a numerical semigroup *S:*

$$\mu(S)\, n(S) \geq g(S) + 1$$

Example

$S = < 5, 9, 13 >$

$\mu(S) = 3$

$g(S) = 21$

$n(S) = 10$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 …

Example

$S = < 5, 9, 13 >$

$\mu(S) = 3$

$g(S) = 21$

$n(S) = 10$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 …

$\mu(S)\, n(S) \geq g(S) + 1$

Example

$S = < 5, 9, 13 >$

$\mu(S) = 3$

$g(S) = 21$

$n(S) = 10$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 …

$\mu(S)\, n(S) \geq g(S) + 1$

$3 \cdot 10 \geq 21 + 1$

Example

$S = \langle 5, 9, 13 \rangle$

$\mu(S) = 3$

$g(S) = 21$

$n(S) = 10$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 …

$\mu(S)\, n(S) \geq g(S) + 1$

$30 \geq 22$

Example

$S = < 5, 9, 13 >$

$\mu(S) = 3$

$g(S) = 21$

$n(S) = 10$

… 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 …

$\mu(S)\, n(S) \geq g(S) + 1$

$30 \geq 22$

Wilf's Conjecture

Cases Proven True –

- S is symmetric
- *S* is pseudosymmetric
- *S* is of maximal embedding dimension

  (maximal embedding dimension means μ(*S*) = the smallest positive element in S)
- μ(*S*) ≤ 3
- g(*S*) ≤ 20
- n(*S*) ≤ 4
- n(*S*) ≥ $\frac{g(S)+1}{4}$

My Investigation

Given –

- All numerical semigroups are the intersection of symmetric and pseudosymmetric numerical semigroups.
- Wilf's Conjecture is proven for symmetric and pseudosymmetric numerical semigroups.

Question –

- Can we relate a numerical semigroup's embedding dimension and number of small elements back to those of parent semigroups?

My Investigation

- Let S_1 and S_2 be numerical semigroups.
- Let $S_3 = S_1 \cap S_2$.

My Investigation

- Let S_1 and S_2 be numerical semigroups.
- Let $S_3 = S_1 \cap S_2$.

Then –

- S_3 is not pseudo/symmetric.
- $n(S_3) \leq \max(\, n(S_1), n(S_2)\,)$.
- If $\mu(S_1) = \mu(S_2) = 2$, then $\mu(S_3) \geq 3$.
- If $\mu(S_1) = \mu(S_2) = 3$, then $\mu(S_3) \geq 3$.

My Investigation

Conjecture –

If $g(S_1) = g(S_2)$, $\mu(S_1) = \mu(S_2)$, and $S_3 = S_1 \cap S_2$,

then $\mu(S_3) \geq \mu(S_1) = \mu(S_2)$

Open Questions

- If the conjecture is true, can it be expanded and used to make more progress on Wilf's conjecture?

- Is there a predictable relationship between the values of n(S_3) and the values of n(S_1) and n(S_2)? If so, can we also utilize this relationship in the investigation of Wilf's conjecture?

Questions

Questions?

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Resources

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