United States Air Force Academy

Intersecting Duals and Ideals of Numerical Semigroups

C1C Jeremy Thompson
United States Air Force Academy
Advisor: Dr. Kurt Herzinger

Summary

- We will further discuss the algebraic structure known as a numerical semigroup and more advanced definitions related to them.

- We will examine intersections of ideals and duals of numerical semigroups.

Overview

- Definitions
- Example
- Structure
- Example
- Sample Proof
- Hypotheses
- Open Questions

Definitions

Numerical Semigroup –

a subset *S* of $\mathbb{N}$ (the non-negative integers) closed under addition, containing zero, and having a largest integer not in *S*

$$S = < 6, 8, 13 >$$

S = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

Definitions

Relative Ideal of S –

a non-empty subset I of $\mathbb{N}$ such that I has a smallest element and if $a \in I$ and $s \in S$, then $a + s \in S$.

$$S = < 6, 8, 13 >$$

$$I = (0, 1)$$

S = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

I = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

Definitions

Dual of I in S –

denoted $S - I$, all integers z such that $z + I \subseteq S$.

$$S = \langle 6, 8, 13 \rangle$$

$$I = (0, 1)$$

$$S - I = (12, 13, 19, 22, 23)$$

S = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

I = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

S - *I* = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24…

Example

$S = < 5, 7, 16 >$

S = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

Example

$S = < 5, 7, 16 >$

$I = (0, 2)$

S = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

I = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

Example

$S = < 5, 7, 16 >$

$I = (0, 2)$

$S - I = (5, 12, 14, 16)$

S = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

I = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

$S - I$ = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

Example

$S = < 5, 7, 16 >$

$I = (0, 2)$

$S - I = (5, 12, 14, 16)$

S = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

I = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

$S - I$ = 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 …

SO… lets get on to the cool stuff already!!!

Structure

Let S_1 and S_2 be numerical semigroups. $S_\cap$ *is* the intersection of S_1 and S_2.

$$S_1 \longrightarrow S_\cap \longleftarrow S_2$$

Structure

I_1, I_2, and $I_\cap$ are the respective ideals of the semigroups, created by the same generating set.

$$\begin{tikzcd}
S_1 \arrow[d] \arrow[dr] & & S_2 \arrow[d] \arrow[dl] \\
I_1 & S_\cap \arrow[d] & I_2 \\
& I_\cap &
\end{tikzcd}$$

Structure

$\cap I$ *is* the intersection of I_1 and I_2.

$$
\begin{tikzcd}
S_1 \arrow[ddd] \arrow[drr] & & & & S_2 \arrow[ddd] \arrow[dll] \\
& & S_\cap \arrow[ddd] & & \\
& & & & \\
I_1 \arrow[ddrr] & & & & I_2 \arrow[ddll] \\
& & & & \\
& & I_\cap \subseteq \cap I & &
\end{tikzcd}
$$

Structure

$S_1 - I_1$, $S_2 - I_2$, $S_\cap - I_\cap$, and $S_\cap - \cap I$ are the duals between the specified semigroups and ideals.

$$
\begin{tikzcd}
S_1 \arrow[drrr] \arrow[dd] & & & & & & S_2 \arrow[dlll] \arrow[dd] \\
& & & S_\cap \arrow[ddl] & & & \\
I_1 \arrow[drrrr] \arrow[dd] & & & & & & I_2 \arrow[dll] \arrow[dd] \\
& & I_\cap \arrow[ddrr] & \subseteq & \cap I \arrow[ddll] & & \\
S_1 - I_1 & & & & & & S_2 - I_2 \\
& & S_\cap - \cap I & \subseteq & S_\cap - I_\cap & &
\end{tikzcd}
$$

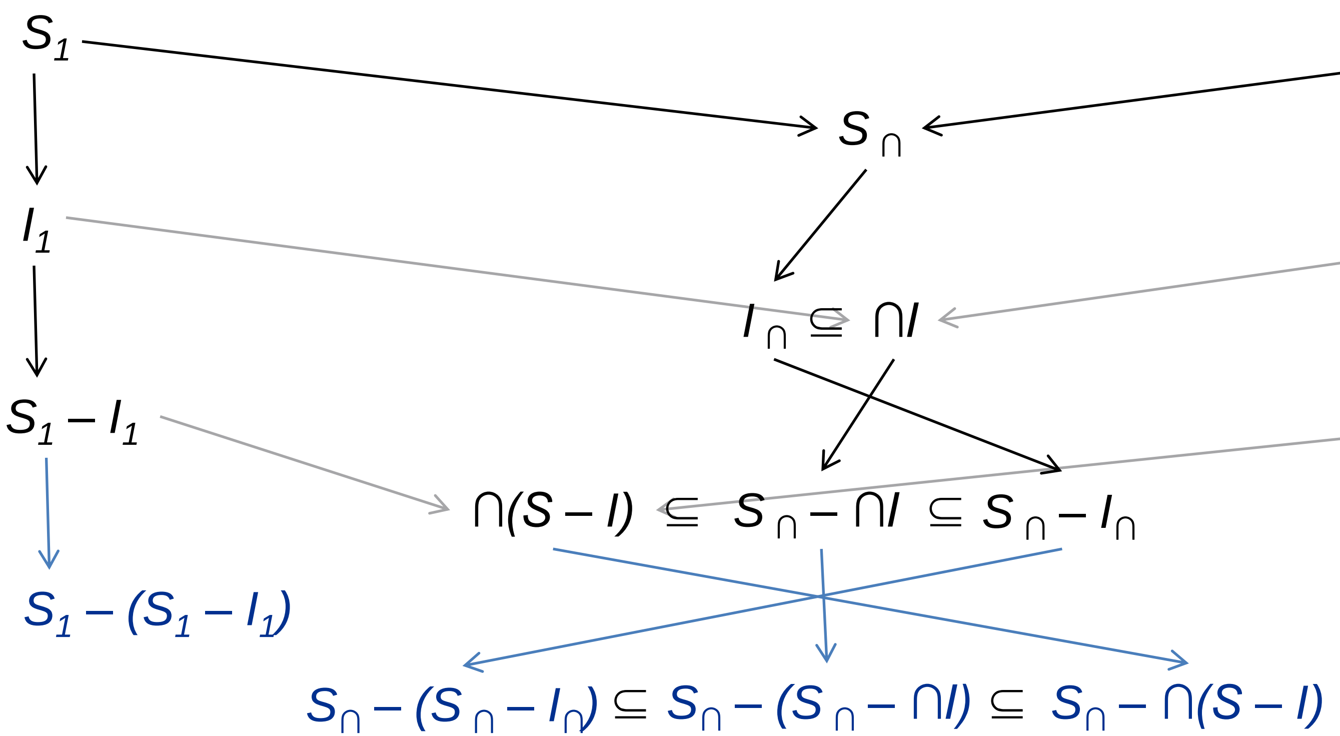
Structure

$\cap(S - I)$ is the intersection of $S_1 - I_1$ and $S_2 - I_2$.

$$\begin{tikzcd}
S_1 \arrow[rrr] \arrow[d] & & & S_\cap \arrow[dl] & & & S_2 \arrow[lll] \arrow[d] \\
I_1 \arrow[d] \arrow[drrrr, gray] & & & & & & I_2 \arrow[d] \arrow[dll, gray] \\
& & I_\cap \arrow[ddrrr] & \subseteq & \cap I \arrow[ddl] & & \\
S_1 - I_1 \arrow[dr, blue] & & & & & & S_2 - I_2 \arrow[dlllll, blue] \\
& \color{blue}{\cap(S - I)} & \subseteq & S_\cap - \cap I & \subseteq & S_\cap - I_\cap &
\end{tikzcd}$$

Structure

$S_1 - (S_1 - I_1)$, $S_2 - (S_2 - I_2)$, $S_\cap - (S_\cap - I_\cap)$, $S_\cap - (S_\cap - \cap I)$, and $S_\cap - \cap(S - I)$ are the duals between the specified semigroups and ideals.



Structure

$\cap(S - \cap(S - I))$ is the intersection of $S_1 - (S_1 - I_1)$ *and* $S_2 - (S_2 - I_2)$.

S_1 $\qquad$ S_2

$S_\cap$

I_1 $\qquad$ I_2

$I_\cap \subseteq \cap I$

$S_1 - I_1$ $\qquad$ $S_2 - I_2$

$\cap(S - I) \subseteq S_\cap - \cap I \subseteq S_\cap - I_\cap$

$S_1 - (S_1 - I_1)$ $\qquad$ $S_2 - (S_2 - I_2)$

$S_\cap - (S_\cap - I_\cap) \subseteq S_\cap - (S_\cap - \cap I) \subseteq S_\cap - \cap(S - I) \subseteq \cap(S - (S - I))$

Structure

$$S_1 \qquad\qquad S_\cap \qquad\qquad S_2$$

$$I_1 \qquad\qquad\qquad\qquad I_2$$

$$I_\cap \subseteq \cap I$$

$$S_1 - I_1 \qquad\qquad\qquad\qquad S_2 - I_2$$

$$\cap(S - I) \subseteq S_\cap - \cap I \subseteq S_\cap - I_\cap$$

$$S_1 - (S_1 - I_1) \qquad\qquad\qquad\qquad S_2 - (S_2 - I_2)$$

$$S_\cap - (S_\cap - I_\cap) \subseteq S_\cap - (S_\cap - \cap I) \subseteq S_\cap - \cap(S - I) \subseteq \cap(S - (S - I))$$

Example

S_1 $\longrightarrow$ $S_\cap$ $\longleftarrow$ S_2

S_1: < 4, 5, 6 > =
0 1 2 3 4 5 6 7 8 9 10 …

S_2: < 5, 6, 7 > =
0 1 2 3 4 5 6 7 8 9 10 …

$S_\cap$:
0 1 2 3 4 5 6 7 8 9 10 …

Example

S_1

< 4, 5, 6 > =

0 1 2 3 4 5 6 7 8 9 10 …

S_2

< 5, 6, 7 > =

0 1 2 3 4 5 6 7 8 9 10 …

$S_\cap$

0 1 2 3 4 5 6 7 8 9 10 …

I_1

(0, 1) =

0 1 2 3 4 5 6 7 8 9 10 …

I_2

(0, 1) =

0 1 2 3 4 5 6 7 8 9 10 …

$I_\cap \subseteq \cap I$

0 1 2 3 4 5 6 7 8 9 10 …   0 1 2 3 4 5 6 7 8 9 10 …

Example

S_1

< 4, 5, 6 > =
0 1 2 3 4 5 6 7 8 9 10 …

S_2

< 5, 6, 7 > =
0 1 2 3 4 5 6 7 8 9 10 …

$S_\cap$

0 1 2 3 4 5 6 7 8 9 10 …

I_1

(0, 1) =
0 1 2 3 4 5 6 7 8 9 10 …

I_2

(0, 1) =
0 1 2 3 4 5 6 7 8 9 10 …

$I_\cap \subseteq \cap I$

0 1 2 3 4 5 6 7 8 9 10 … 0 1 2 3 4 5 6 7 8 9 10 …

$S_1 - I_1$

0 1 2 3 4 5 6 7 8 9 10 …

$S_2 - I_2$

0 1 2 3 4 5 6 7 8 9 10 …

$\cap(S - I) \subseteq S_\cap - \cap I \subseteq S_\cap - I_\cap$

0 1 2 3 4 5 6 7 8 9 10 … 0 1 2 3 4 5 6 7 8 9 10 … 0 1 2 3 4 5 6 7 8 9 10 …

Example

S_1

< 4, 5, 6 > =
0 1 2 3 4 5 6 7 8 9 10 …

$S_\cap$

0 1 2 3 4 5 6 7 8 9 10 …

S_2

< 5, 6, 7 > =
0 1 2 3 4 5 6 7 8 9 10 …

I_1

(0, 1) =
0 1 2 3 4 5 6 7 8 9 10 …

$I_\cap \subseteq \cap I$

0 1 2 3 4 5 6 7 8 9 10 …   0 1 2 3 4 5 6 7 8 9 10 …

I_2

(0, 1) =
0 1 2 3 4 5 6 7 8 9 10 …

$S_1 - I_1$

0 1 2 3 4 5 6 7 8 9 10 …

$S_2 - I_2$

0 1 2 3 4 5 6 7 8 9 10 …

$\cap(S - I) \subseteq S_\cap - \cap I \subseteq S_\cap - I_\cap$

0 1 2 3 4 5 6 7 8 9 10 … 0 1 2 3 4 5 6 7 8 9 10 … 0 1 2 3 4 5 6 7 8 9 10 …

Sample Proof

There are two types of proofs inside this structure:

- Ideal Proofs
  - The intersections are ideals of $S_{\cap}$
- Containment Proofs
  - The ideals of $S_{\cap}$ in respective levels have subset relationships

Sample Proof

S_1

$S_\cap$

I_1

$I_\cap \subseteq \cap I$

$S_1 - I_1$

$\cap(S - I) \subseteq S_\cap - \cap I \subseteq S_\cap - I_\cap$

$S_1 - (S_1 - I_1)$

$S_2 - (S_2 - I_2)$

$S_\cap - (S_\cap - I_\cap) \subseteq S_\cap - (S_\cap - \cap I) \subseteq S_\cap - \cap(S - I) \subseteq \cap(S - (S - I))$

Sample Proof

- Ideal Proofs
  - Depend upon elements of the intersections being related to all elements in $S_{\cap}$

- Containment Proofs
  - Most depend upon the following theorem:
  - If there are relative ideals I, J such that $I \subseteq J$, then $S - J \subseteq S - I$.
  - Difficult to prove intersections are contained in all other ideals

Sample Proof

Containment Proof –

$I_{\cap} \subseteq \cap I$

Let $i \in I_{\cap}$. By the definition of an ideal we may say $i = k + s_{\cap}$, where k is a generator of the ideal and $s_{\cap} \in S_{\cap}$. Because $S_{\cap}$ is the intersection of S_1 and S_2, we may also say $i = k + s_1 = k + s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$. Therefore $i \in I_1, I_2$ and $i \in \cap I$. Thus $I_{\cap} \subseteq \cap I$. ∎

Hypotheses

- $\cap(S - I) \subseteq S_{\cap} - \cap I$, $\cap(S - (S - I)) \subseteq S_{\cap} - (S_{\cap} - I_{\cap})$

- Structure generalizes easily to more semigroups, provided the number is finite

Open Questions

- What can be said about the properties of the intersections (Semigroups, Relative Ideas, and Duals) based upon the properties of their parent structures?

- What is $S_{\cap} - \cap(S - (S - I))$?

Questions?

C09Jeremy.Thompson@USAFA.edu


UNITED STATES AIR FORCE ACADEMY